Extended local balance model of turbulence and Couette-Taylor flow

A. M. Balonishnikov*
Department of Natural Sciences and Ecology, St. Petersburg State Academy for Economics and Engineering, 27 Marata, 197002 St. Petersburg, Russia

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An extended local balance model of turbulence, based on a new transport equation for the dissipation rate with a negative diffusion coefficient, is presented. Analytical solutions for the mean velocity and the dissipation rate for the turbulent Couette-Taylor problem are derived. The dependence of torque on the Reynolds number is obtained. These solutions depend only on two constants $k=0.4$ and $C=9.5$ of the turbulent boundary layer and, within the limits of a narrow channel, are reduced to the well-known von Kármán's solutions for planar Couette flow. Strange attractor behavior in this limit is also observed.

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I. INTRODUCTION

The existing models of fully developed turbulence can be classified as large-scale models and small-scale models. Small-scale models deal with eddies smaller than the integral scale of turbulence (large eddy simulation in the Smagorinsky type of models). Subgrid Reynolds stresses depend essentially on the spatial grid size. Large-scale models determine the integral scale of turbulence by means of its independent variables (or its derivatives). In recent years, there has been a great deal of interest in the possible connection between fully developed turbulence and deterministic chaos. There is evidence [1] that for some types of turbulent flow there is a chaotic attractor with a dimension about 10 even for large Reynolds numbers. It may be expected that large-scale models demonstrate such chaotic behavior, but classical large-scale models such as the ''k-ε model,'' ''k-ω model,'' and models of Reynolds stresses transport give only an extremely smooth distribution of such variables as mean velocity, kinetic energy, and dissipation rate in space and in time. Thus, the creation of new models with chaotic temporary (at least) and realistic spatial distributions are very desirable in theoretical as well as in practical aspects (propagation of waves through turbulent medium, weather forecast).

Investigations of deterministic chaos in extended systems are mostly carried out based on the Kuramoto-Sivashinsky equation [2,3]. The essential dependence on the initial conditions is provided by a negative coefficient of diffusion. Regularization of such ill-posed mathematical problems is obtained by adding the operator Δ^2 (Δ denotes Laplace's operator) in these equations. A model with a negative coefficient of turbulent viscosity was considered in [4], and a hypothesis of ''a negative diffusion'' of vorticity was proposed in [5]. They stimulated us to present an alternative model of turbulence.

II. TURBULENCE MODEL

A hypothesis of the negative diffusion of the dissipation rate and an extended local-balance model were put forward in [6] and developed in [7,8],

$$\partial_t u_i + u_j \partial_j u_i = -\partial p^* + \partial_j(\nu_T \partial_j u_i) - \alpha \partial^2 (D_T \partial^2 u_i), \quad (1)$$

$$\partial_t \varepsilon + u_j \partial_j \varepsilon = \beta[\kappa \varepsilon S - \kappa^{-1} \partial_j(\nu_T \partial_j \varepsilon)] - \gamma \partial^2 (D_T \partial^2 \varepsilon), \quad (2)$$

$$\partial_i u_i = 0, \quad (3)$$

where u_i is the mean, or large-scale, velocity ($i=1,2,3$), $\varepsilon = \frac{1}{2}\langle \nu(\partial_j u_i' + \partial_i u_j')^2 \rangle$ denotes the dissipation rate, $\partial^2 = \partial_i \partial_i$ is the sum over i, $p^* = \varrho^{-1} p + \langle u_i' u_i' \rangle /3$ is the modified pressure, α, β, γ denote constants of the model, κ is the von Kármán constant, S is the invariant of the strain rate tensor $S = \sqrt{2 S_{ij} S_{ji}}$ with $S_{ij} = 1/2(\partial_j u_i + \partial_i u_j)$, $\nu_T = \varepsilon / S^2$ is a coefficient of turbulent viscosity, and $D_T = \varepsilon^2 / S^5$ is a coefficient of ''superviscosity,'' derived from dimensional analysis.

Very simple but unusual assumptions are made. It is assumed that the transport equation of the dissipation rate of turbulent energy depends only on the dissipation rate itself, the invariant S, and their spatial derivatives. A conservative form of the equation is postulated. The first linear term on the right-hand side of Eq. (2) provides the growth of the dissipation rate. It corresponds to the growth of small-scale vorticity under strong shear strain [9] and this term is similar to linear terms in the well-known Saffman k-w model [10]. In order to compensate this source term in Eq. (2), the second term must be included with a negative coefficient of diffusion. It can be easily seen in a logarithmic boundary layer: $\varepsilon = u^{*3}/(\kappa x)$, $S = u^*/(\kappa x)$, where u^* is the skin friction velocity and x is the distance from a wall. This situation is different from the well-known k-ε model [11], where the diffusion of the dissipation rate is the usual diffusion with a positive coefficient of diffusion. The third term on the right-hand side of Eq. (2) cuts off small-scale components of the dissipation rate and limits the growth of the dissipation rate due to the cube of the dissipation rate outside of the logarithmic boundary layer. This regularizing term is equal to zero inside the logarithmic boundary layer, where Reynolds stresses are constants. The expression for turbulent viscosity $\nu_T = \varepsilon / S^2$ can be readily derived from the approximation of the turbulent energy local balance: production is equal to dissipation in the equation of balance of turbulent energy [12]:

$$\langle u_i' u_j' \rangle \partial_j U_i = -\nu \langle \partial_k u_i' \partial_k u_i' \rangle \quad (4)$$

*Electronic address: seie@comp.engec.spb.ru

or

$$-\langle u_i' u_j' \rangle \partial_j U_i = \varepsilon. \tag{5}$$

Indeed, if we set

$$-\langle u_i' u_j' \rangle = 2\nu_T S_{ij} + \delta_{ij} \langle u_i' u_i' \rangle /3 \tag{6}$$

and substitute this expression in the equation of the local turbulent energy balance, then turbulent energy comes out from this equation due to the incompressibility condition. It was done for the first time in [13]. In this manner we obtain the expression for the coefficient of turbulent viscosity, which is written above.

Boundary conditions are determined as in the standard "k-ε model" by the wall functions $u = u_* \kappa^{-1} \ln C u_* x/\nu$ and $\varepsilon = u_*^3/(\kappa x)$, $x \to 0$, where x is the distance from the wall and u_* is the skin friction velocity.

The first term on the right-hand side in the dissipation rate transport equation can be explained by the total increase of small-scale vorticity under strong large-scale shear. The second term compensates for the first in the logarithmic turbulent boundary layer ("negative diffusion" of dissipation appears in a formal way). Terms in both evolutionary equations for u and ε with spatial derivatives of fourth order do not eliminate the existence of the logarithmic layer in the model, but they do cut off small-scale modes and limit the growth of the unstable integral scale-size modes of large-scale velocity and dissipation.

Note that the model applied to the plane turbulent Couette flow possesses a self-regularizing property (being considered on a large-scale spatial grid). The stationary solution, coinciding with von Kármán solution [14], is stable [7].

III. TURBULENT COUETTE-TAYLOR FLOW

Let us consider turbulent flow between two contrarotating cylinders (or when one of them is at rest). v_1 denotes the rotational velocity of the inner cylinder, v_2 is the rotational velocity of the outer cylinder, and the radii of the cylinders are a,b $(b>a)$. Angular velocities are $\omega_1 = v_1/a$ and $\omega_2 = v_2/b$. Let the inner cylinder rotate in a counterclockwise direction (or be at rest) and the outer cylinder rotate in a clockwise direction (or be at rest). Boundary conditions are defined as $u = v_1 - u_{\rm in}^* \kappa^{-1} \ln C(r-a) u_{\rm in}^*/\nu$ if $r \to a$, $u = -v_2 + u_{\rm out}^* \kappa^{-1} \ln C(b-r) u_{\rm out}^*/\nu$ if $r \to b$, $\varepsilon = u_{\rm in}^{*3} \kappa^{-1} (r-a)^{-1}$ if $r \to a$, $\varepsilon = u_{\rm out}^{*3} \kappa^{-1} (b-r)^{-1}$ if $r \to b$, r is the distance from the axis of the cylinders, $u_{\rm in}^*$ is the skin friction velocity at the inner cylinder, and $u_{\rm out}^*$ is the skin friction at the outer cylinder. For turbulent Taylor-Couette flow, as it is well known, turbulent stresses τ vary with channel width as

$$\tau r^2 = \tau_1 r_1^2 = \tau_2 r_2^2,$$

where $\tau_1 = u_{\rm in}^{*2}$ and $\tau_2 = u_{\rm out}^{*2}$.

As we limit ourselves to a stationary problem, the regularizing spatial derivatives of fourth order will be omitted. Then we shall seek the solutions $u = u(r)$ and $\varepsilon = \varepsilon(r)$. The dissipation rate transport equation in the cylindrical coordinates will be

$$\kappa \varepsilon S - \kappa^{-1} r^{-1} \partial_r (r \varepsilon S^{-2} \partial_r \varepsilon) = 0. \tag{7}$$

Taking into account the local balance assumption (5), we have $\tau S = \varepsilon$ (τ is known), and excluding S, the equation for the dissipation will be reduced to a nonlinear ordinary differential equation for ε:

$$\varepsilon^2 r^2 \tau_1^{-1} r_1^{-2} - \kappa^{-2} r^{-1} \partial_r (r^{-3} \tau_1^{-2} r_1^4 \varepsilon^{-1} \partial_r \varepsilon) = 0. \tag{8}$$

This equation is readily reduced to standard form by the substitution $q = r^4$, and returning to the initial variable r we shall have a unique solution satisfying the boundary conditions [6,8]:

$$\varepsilon = 4 a^3 u_{\rm in}^{*3} \pi \kappa^{-1} (b^4 - a^4)^{-1} \csc[\pi (r^4 - a^4)/(b^4 - a^4)]. \tag{9}$$

We shall have for the cylinder geometry $S = -\partial_r (u/r)$ for the chosen direction of the cylinders's rotations. Due to $S = \varepsilon/\tau$, the resulting equation for the mean velocity is

$$-r \partial_r (u/r) = 4\pi r^2 a u_{\rm in}^* \kappa^{-1} (b^4 - a^4)^{-1} \times \csc[\pi (r^4 - a^4)/(b^4 - a^4)]. \tag{10}$$

After integration over the gap, the mean velocity is

$$u = -4 a r u_{\rm in}^* \pi \kappa^{-1} (b^4 - a^4)^{-1} \times \int_p^r R \csc[\pi (R^4 - a^4)/(b^4 - a^4)] dR + \omega_0 r, \tag{11}$$

where ω_0 is an integration constant, p is an arbitrary point, and $a < p < b$. The integral has singularities at $r = a$ and $r = b$, thus it is convenient to regularize the integral by subtracting the corresponding singularities. Values ω_0 and $u_{\rm in}^*$ can be obtained by making the solution for u fit the logarithmic wall functions at $r = a$ and $r = b$. The arbitrary constant p is eliminated in the final formula, and a nonlinear algebraic equation at this stage (a resistance law such as that of von Kármán–Prandtl) is derived for $u_{\rm in}^*$. Omitting evident algebraic transformations, the final results are

$$u(q)/(v_1 q) = -4 z \kappa^{-1} \pi (Q^4 - 1)^{-1} J_1 - z \kappa^{-1} \ln[C R z (q-1)/(Q-1)] + 1, \tag{12}$$

where

$$J_1(q,Q) = \int_1^q (x \{ \csc[\pi (x^4 - 1)/(Q^4 - 1)] - (Q^4 - 1)(4\pi)^{-1} (x-1) \}) dx. \tag{13}$$

Here, $R = v_1 (b-a)/\nu$ is the Reynolds number, $q = r/a$, $Q = b/a$, and $z = u_{\rm in}^*/v_1$ (dimensionless skin friction velocity) is calculated from the nonlinear equation (resistance law)

$$4 Q u_{\rm in}^* \pi \kappa^{-1} (Q^4 - 1)^{-1} J_0 + u_{\rm in}^* Q \kappa^{-1} \ln[C u_{\rm in}^* (b-a)/\nu] + u_{\rm out}^* \kappa^{-1} \ln[C u_{\rm out}^* (b-a)/\nu] = v_2 + v_1 Q, \tag{14}$$

TABLE I. Dependence of experimental [15] and theoretical (model presented) dimensionless torque G on the Reynolds number (inner cylinder rotates, outer cylinder is at rest), radius ratio $a/b=0.7253$.

R	$G_{\rm expt}$	$G_{\rm theory}$
10^3	4.0×10^5	1.0×10^5
10^4	8.4×10^6	5.4×10^6
10^5	4.2×10^8	3.3×10^8
10^6	2.7×10^{10}	2.2×10^{10}

where $J_0=\int_1^Q dx\{x\csc[\pi(x^4-1)/(Q^4-1)]-(Q^4-1)(4\pi)^{-1}(x-1)^{-1}-(Q^4-1)(4\pi)^{-1}Q^{-2}(Q-x)^{-1}\}$.

It can be shown that the solution is reduced to a von Kármán solution [14] for plane Couette flow in the narrow-gap limit [8]:

$$u=u^*\kappa^{-1}\ln\tan[x\pi/(4h)],$$

$$\varepsilon=u^{*3}\kappa^{-1}(2h)^{-1}\csc[x\pi/(2h)],$$

where $2h$ is the channel width and x is the distance from a wall.

IV. COMPARISON WITH EXPERIMENTS

The theoretical dependence of the torque G on the Reynolds number R for turbulent Couette-Taylor flow and experimental data [15] are presented in Table I, where $G=c_fR^2$ and the friction coefficient $c_f=2\pi(u_1^*/v_1)^2(b/a-1)^{-2}$.

Theoretical (this model) and experimental resistance laws [16,17] are presented in Tables II and III.

The theoretical dependence of the mean velocity on the distance from an inner cylinder for turbulent Couette-Taylor flow and experimental data [18] are presented in Table IV. There is an agreement between the model and experimental data at high Reynolds numbers.

There are several competing models. The Saffman model being applied to Couette-Taylor flow had to be solved numerically, and the solution depended on three constants of the model [19]. Special models for turbulent Couette-Taylor flow were put forward [15,20,21]. Besides von Kármán–Prandtl constants, solutions from these models include an additional constant or constants. For example, a simple model [15] with two matched logarithmic layers depends on the distance (in dimensionless form) at which these layers are matched. It was chosen in the middle of the gap. It is evident if the gap is narrow, but where is the matching point that should be chosen in the case of the wide gap and at arbitrary rotating velocities of cylinders?

TABLE II. Experimental [16] and theoretical (model presented) resistance laws, $\zeta=\log_{10}(2\tau_1\times10^3/\rho v_1^2)$, radius ratio $a/b=0.685$, Reynolds number $R_1=v_1(b-a)/\nu$ (inner cylinder rotates, outer cylinder is at rest).

$\log_{10}R_1$	$\zeta_{\rm expt}$	$\zeta_{\rm theory}$
2.5	1.01	0.87
3.0	0.72	0.69
3.5	0.46	0.51
4.0	0.29	0.38
4.5	0.18	0.27

TABLE III. Experimental [17] and theoretical (model presented) resistance laws, $\zeta=\log_{10}(\tau_2/\rho v_2^2)$, radius ratio $a/b=0.75$, Reynolds number $R_2=v_2(b-a)/\nu$ (inner cylinder is at rest, outer cylinder rotates).

$\log_{10}R_1$	$\zeta_{\rm expt}$	$\zeta_{\rm theory}$
4.04	−3.27	−3.46
4.36	−3.52	−3.51
4.53	−3.50	−3.55
4.64	−3.53	−3.58
4.75	−3.54	−3.59

Since solution (14) does not include constants besides $\kappa=0.4$ and $C=9.5$, it can be applied for arbitrary turbulent Couette-Taylor flow (except for corotating cylinders, where the shear strain rate is not large enough). Thus, the presented generalized von Kármán solutions can be used as a test for future experiments in Couette-Taylor flow at very large Reynolds numbers. Hence, the generalized von Kármáan solutions have obvious advantages over those mentioned above.

V. LARGE-SCALE MODELING OF NONSTATIONARY PLANE COUETTE FLOW

Let large-scale velocity $\mathbf{u}=(0,u(x,t),0)$, $\varepsilon(x,t)$ is dissipation rate, x is the distance from a wall, and another wall is moving in a direction parallel to the first one with relative velocity $2V_C$. Equations of the model are reduced to

$$\partial_t u=\partial_x(\varepsilon/\partial_x u)-\alpha\partial_x^2\{[\varepsilon^2/(\partial_x u)^5]\partial_x^2 u\},\qquad(15)$$

$$\partial_t\varepsilon=\beta\{\varepsilon|\partial_x u|-\kappa^{-2}\partial_x[\varepsilon/(\partial_x u)^2\partial_x\varepsilon]\}-\gamma\{\partial_x^2[\varepsilon^2/(\partial_x u)^5\partial_x^2\varepsilon]\}.\qquad(16)$$

If regularizing constants $\alpha=0$ and $\gamma=0$, the system of the equations has the solutions [8]

$$u=u^*\kappa^{-1}\ln\tan(\pi x/4h),\qquad(17)$$

$$\varepsilon=\pi u^{*3}/(2\kappa h)\csc(\pi x/2h).\qquad(18)$$

TABLE IV. Dependence of experimental [18] and theoretical (model presented) mean velocity on cylinder gap width, Reynolds number $R=5.03\times10^4$, radius ratio $a/b=2/3$ (inner cylinder rotates, outer cylinder is at rest).

$(r\text{-}a)/(b\text{-}a)$	$U/U_{\rm 1expt}$	$U/U_{\rm 1theory}$
0.187	0.448	0.43
0.318	0.425	0.41
0.449	0.405	0.40
0.579	0.391	0.39
0.708	0.373	0.38
0.841	0.356	0.36
0.907	0.345	0.34

These expressions are well-known von Kármán solutions for turbulent plane Couette flow [14]. The k-ε model and the Reynolds stresses transport model give the same analytical solutions [22]. It is doubtful that the k-ε model would give analytical solutions for turbulent Couette-Taylor flow because equations of the model should be modified with additional terms for rotating flows [23]. The requirement for u to become the logarithmic wall function at the wall gives a resistance law

$$V_C/u^* = \kappa^{-1}\ln[4CRu^*/(\pi V_C)], \tag{19}$$

where $R = V_C h/\nu$ is the Reynolds number. It can be shown that there is a good agreement of this law with experiments of different authors at sufficiently large Reynolds numbers [8]. Large-scale nonstationary modeling is assumed to be valid at $\alpha \neq 0, \gamma \neq 0$. For better resolution of domains near the moving walls, a homogenization of the problem

$$p = \ln\tan(x\pi/4h) \tag{20}$$

is used. This is an extension of the Biot–Kármán transformation for mean velocity in the turbulent boundary layer. It is convenient to introduce new variables:

$$D(p,t) = \varepsilon(p,t)/\varepsilon_0(p),$$

$$F(p,t) = u(p,t) - u_0(p),$$

where $u_0 = u^* p \kappa^{-1}$ and $\varepsilon_0 = u^{*3}\pi\cosh(p)/(2h\kappa)$ are von Kármán solution in new p coordinates. Boundary conditions for the nonstationary problem are imposed at $p = \pm p^*, p^* = \ln\tan(\pi l^*/4h)$, where $l^* = h/u^*$, l^* and p^* is the skin friction lengths in x and p coordinates. These boundary conditions are chosen as $D(\pm p^*,t) = 1$, $\partial_p D(\pm p^*,t) = 0$, $F(\pm p^*,t) = 0$, $\partial_p F(\pm p,t) = 0$.

Equations of the model with the new variable p are

$$\begin{aligned}\partial_t F = &(\pi/2)\cosh(p)\partial_p[D/(1+\kappa\partial_p F)] \\ &- \alpha_1(\pi/2)\cosh(p)\partial_p(\cosh(p)\partial_p \\ &\times\{D^2(1+\kappa\partial_p F)^{-5}/\cosh^2(p)\partial_p[\cosh(p)\partial_p F]\}),\end{aligned} \tag{21}$$

$$\begin{aligned}\partial_t D = &\alpha_0[\cosh(p)D|1+\kappa\partial_p F|] \\ &- \partial_p\{D(1+\kappa\partial_p F)^{-2}\partial_p[\cosh(p)D]\} \\ &- \alpha_2\kappa^3\partial_p(\cosh(p)\partial_p[D^2(1+\kappa\partial_p F)^{-5}/\cosh^2(p)] \\ &\times\partial_p\{\cosh(p)\partial_p[\cosh(p)(D-1)]\}).\end{aligned} \tag{22}$$

To make regularizing operators less time-consuming, these are chosen in a slightly different form,

$$\begin{aligned}\partial_t F = &(\pi/2)\cosh(p)\partial_p[D/(1+\kappa\partial_p F) \\ &- \alpha_1\cosh(p)\partial^2_{pp}(D^2\partial^2_{pp}F)],\end{aligned} \tag{23}$$

$$\begin{aligned}\partial_t D = &\alpha_0[\cosh(p)D|1+\kappa\partial_p F|] - \partial_p[D(1+\kappa\partial_p F)^{-2}] \\ &\times\partial_p[\cosh(pD)] - \alpha_2\cosh(p)\partial^2_{pp}(D^2\partial^2_{pp}D).\end{aligned} \tag{24}$$

Coefficient $\cosh(p)$ in these equations characterizes the growth rate of the integral turbulent scale in the direction from a wall to the center of the channel. It results in a stiffness of the equations and makes it necessary to apply special methods to solve the problem of long-term behavior. The system of equation was solved by method of lines, where spatial derivatives were approximated by symmetric finite differences. The obtained system of stiff ODE was solved by the modified Gear method from Fortran subroutine library NAG-8. The Reynolds number R was chosen $R = 25R_{Cr}$, here $R_{Cr} = 118$ according to [24], where experimental chaos in circular Couette flow was observed. The number of internal p nodes was chosen to be $N = 6$ or $N = 7$ (system of ODE with 12 or 14 equations). If $N < 6$, stationary state, the von Kármán solution is linearly stable (''self-regularization''). Available computers did not permit us to investigate the case $N \to \infty$, which is more interesting from a theoretical point of view because the possibility exists of spatio-temporal chaos. Method [25] was chosen to calculate the largest Lyapunov exponent λ_1. The results are (a) $\lambda_1 = 0.0016$, $N = 6, \alpha_0 = 1.0, \alpha_1 = 0.1, \alpha_2 = 0.1$; (b) $\lambda_1 = 1.0, N = 6, \alpha_0 = 1.0, \alpha_1 = 0.1, \alpha_2 = 0.1$; (c) $\lambda_1 = 0.006, N = 7$, $\alpha_0 = 1.0, \alpha_1 = 0.2, \alpha_2 = 0.2$. Since solutions are bounded, the positive largest Lyapunov exponents are evidence of the deterministic chaos. A power spectrum of the velocity in the computational experiment (a) had a double broadband component, and in case (c) a broadband component was not observed. In general we can see only about a qualitative agreement with experiment [24].

VI. CONCLUSIONS

There is an agreement between the presented model and Couette-Taylor experiments [15–18] about the resistance law and mean velocity at sufficiently large Reynolds numbers. It should be mentioned that the model gives a good agreement also with experimental data [26] and bad agreement with resistance law [27], as it was shown in [8]. The proposed model agrees completely with the Kolmogorov picture of turbulent cascade [28], because the dissipation rate is the only additional large-scale parameter. Moreover, the fluctuation in the dissipation rate according to the refined Kolmogorov hypothesis [29] can be explained by the deterministic chaos in the dynamics of large-scale variables of the model. Application of the presented ''ε model'' of turbulence to the flows in pipes and to the decay of the turbulence is controversial, and now this point is under investigation. It is necessary to determine values of constants of the model also. The model gives two possibilities in the description of fully developed turbulence: Langevin equations with random forces, or a ''strange-attractor'' description with regularizing operators (or their combination). From the available experimental data [30], it seems that in turbulent Couette-Taylor flow we deal with ''strange-attractor'' behavior.

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